

CANONICAL WEIERSTRASS REPRESENTATION OF MINIMAL SURFACES IN EUCLIDEAN SPACE

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ABSTRACT. Using the fact that any minimal strongly regular surface carries locally canonical principal parameters, we obtain a canonical representation of these surfaces, which makes more precise the Weierstrass representation in canonical principal parameters. This allows us to describe locally the solutions of the natural partial differential equation of minimal surfaces.

1. INTRODUCTION

In [1] we proved that any minimal strongly regular surface can be endowed locally with canonical principal parameters. Using this result, in this note we prove the following

Theorem 1. (Canonical Weierstrass representation) *Any minimal strongly regular surface $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y)$, $(x, y) \in \mathcal{D} \subset \mathbb{C}$ parameterized with canonical principal parameters has locally a representation of the type*

$$\begin{aligned} z_1 &= \Re \left(\frac{1}{2} \int_{\mathbf{z}_0}^{\mathbf{z}} \frac{w^2 - 1}{w'} dz \right), \\ \mathcal{M} : \quad z_2 &= \Re \left(-\frac{i}{2} \int_{\mathbf{z}_0}^{\mathbf{z}} \frac{w^2 + 1}{w'} dz \right), \\ z_3 &= \Re \left(- \int_{\mathbf{z}_0}^{\mathbf{z}} \frac{w}{w'} dz \right), \end{aligned}$$

where

$$w = u(x, y) + iv(x, y), \quad \mu := \frac{(u_x^2 + u_y^2)}{(u^2 + v^2 + 1)^2}$$

is a holomorphic function satisfying the conditions

$$\mu > 0, \quad \mu_x \mu_y \neq 0.$$

As an application of this theorem we obtain a local description of the solutions of the natural partial differential equation of minimal surfaces.

2. PRELIMINARIES

Let $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y)$, $(x, y) \in \mathcal{D}$ be a surface in Euclidean space \mathbb{R}^3 and ∇ be the flat Levi-Civita connection of the standard metric in \mathbb{R}^3 . The unit normal vector field to \mathcal{M} is denoted by l and E, F, G ; e, f, g stand for the coefficients of the first and the second fundamental forms, respectively. All functions are supposed to be in the class \mathcal{C}^∞ .

The considerations in this note are local.

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We suppose that the surface has no umbilical points and the principal lines on \mathcal{M} form a parametric net, i.e.

$$F(x, y) = f(x, y) = 0, \quad (x, y) \in \mathcal{D}.$$

Then the principal curvatures ν_1, ν_2 and the principal geodesic curvatures (the geodesic curvatures of the principal lines) γ_1, γ_2 are given by

$$\nu_1 = \frac{e}{E}, \quad \nu_2 = \frac{g}{G}; \quad \gamma_1 = -\frac{E_y}{2E\sqrt{G}}, \quad \gamma_2 = \frac{G_x}{2G\sqrt{E}}.$$

We consider the tangential frame field $\{X, Y\}$ determined by

$$X := \frac{\mathbf{z}_x}{\sqrt{E}}, \quad Y := \frac{\mathbf{z}_y}{\sqrt{G}}$$

and assume that the moving frame XYl is always right oriented so that $\nu_1 - \nu_2 > 0$. The following Frenet type formulas for the frame field XYl are valid

$$(2.1) \quad \begin{aligned} \nabla_X X &= \gamma_1 Y + \nu_1 l, & \nabla_Y X &= \gamma_2 Y, \\ \nabla_X Y &= -\gamma_1 X, & \nabla_Y Y &= -\gamma_2 X + \nu_2 l, \\ \nabla_X l &= -\nu_1 X, & \nabla_Y l &= -\nu_2 Y. \end{aligned}$$

The Codazzi equations are as follows

$$(2.2) \quad \gamma_1 = \frac{Y(\nu_1)}{\nu_1 - \nu_2}, \quad \gamma_2 = \frac{X(\nu_2)}{\nu_1 - \nu_2},$$

A surface $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y)$, $(x, y) \in \mathcal{D}$ parameterized with principal parameters is *strongly regular* [1] if

$$\gamma_1(x, y)\gamma_2(x, y) \neq 0, \quad (x, y) \in \mathcal{D}.$$

Because of (2.2)

$$\gamma_1\gamma_2 \neq 0 \iff (\nu_1)_y(\nu_2)_x \neq 0.$$

Now let \mathcal{M} be a minimal strongly regular surface, whose parametric net is principal. We use the following notations:

$$\nu := \nu_1 > 0, \quad \nu_2 = -\nu < 0, \quad \nu_1 - \nu_2 = 2\nu > 0$$

and refer to the function ν as the normal curvature function.

In [1] we proved that any minimal strongly regular surface \mathcal{M} admits locally canonical principal parameters (x, y) so that the coefficients E, G and e, g are given by:

$$(2.3) \quad E = G = \frac{1}{\nu} > 0, \quad e = -g = 1.$$

Remark 2.1. Here we use different from [1] normalization for E and G in order to obtain more appropriate form for the canonical representation of minimal surfaces.

Further we assume, that the minimal strongly regular surface $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y)$, $(x, y) \in \mathcal{D}$ is parameterized with canonical principal parameters. Then the principal geodesic curvatures are given by

$$(2.4) \quad \gamma_1 = (\sqrt{\nu})_y, \quad \gamma_2 = -(\sqrt{\nu})_x.$$

The fundamental Bonnet theorem, applied to minimal strongly regular surfaces parameterized with canonical principal parameters states as follows (cf [1]):

Bonnet theorem for minimal surfaces in canonical principal parameters. *Given a function $\nu(x, y) > 0$ in a neighborhood \mathcal{D} of (x_0, y_0) with $\nu_x \nu_y \neq 0$, satisfying the equation*

$$(2.5) \quad \Delta \ln \nu + 2\nu = 0$$

and an initial right oriented orthonormal frame $\mathbf{z}_0 X_0 Y_0 l_0$.

Then there exists a unique minimal strongly regular surface $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y), (x, y) \in \mathcal{D}_0 ((x_0, y_0) \in \mathcal{D}_0 \subset \mathcal{D})$, such that

- (i) *(x, y) are canonical principal parameters;*
- (ii) *$\mathbf{z}(x_0, y_0) = \mathbf{z}_0, X(x_0, y_0) = X_0, Y(x_0, y_0) = Y_0, l(x_0, y_0) = l_0$;*
- (iii) *the invariants of \mathcal{M} are the following functions*

$$\nu_1 = \nu, \quad \nu_2 = -\nu, \quad \gamma_1 = (\sqrt{\nu})_y, \quad \gamma_2 = -(\sqrt{\nu})_x.$$

Further we refer to (2.5) as the natural partial differential equation of minimal surfaces.

The above statement gives a one-to-one correspondence between minimal strongly regular surfaces (considered up to a motion) and the solutions of the natural partial differential equation, satisfying the conditions

$$(2.6) \quad \nu > 0, \quad \nu_x \nu_y \neq 0.$$

According to [1] the sign of the function $\nu_x \nu_y$ divides the class of minimal strongly regular surfaces into two geometric subclasses (invariant with respect to motions and changes of parameters). Any reflection of \mathbb{R}^3 with respect to a plane transforms each of these subclasses onto the other.

3. CANONICAL REPRESENTATION OF MINIMAL STRONGLY REGULAR SURFACES

Let $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y), (x, y) \in \mathcal{D}$ be a minimal strongly regular surface parameterized with canonical principal parameters. In view of (2.3) and (2.4) formulas (2.1) become

$$(3.1) \quad \begin{aligned} \nabla_X X &= (\sqrt{\nu})_y Y + \nu l, \\ \nabla_X Y &= -(\sqrt{\nu})_y X, \\ \nabla_X l &= -\nu X; \\ \nabla_Y X &= -(\sqrt{\nu})_x Y, \\ \nabla_Y Y &= (\sqrt{\nu})_x X - \nu l, \\ \nabla_Y l &= \nu Y \end{aligned}$$

and the integrability conditions for (3.1) reduce to (2.5).

Equalities (3.1) imply the following formulas for the Gauss map $l = l(x, y)$:

$$(3.2) \quad \begin{aligned} l_{xx} &= \frac{\nu_x}{2\nu} l_x - \frac{\nu_y}{2\nu} l_y - \nu l, \\ l_{xy} &= \frac{\nu_y}{2\nu} l_x + \frac{\nu_x}{2\nu} l_y, \\ l_{yy} &= -\frac{\nu_x}{2\nu} l_x + \frac{\nu_y}{2\nu} l_y - \nu l \end{aligned}$$

and the unit normal vector function $l(x, y)$ satisfies the differential equation:

$$\Delta l + 2l = 0.$$

The next statement makes precise the relation between the properties of the Gauss map of a minimal surface and the canonical principal parameters.

Proposition 3.1. *Let $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y)$, $(x, y) \in \mathcal{D}$ be a minimal strongly regular surface parameterized with canonical principal parameters. Then the Gauss map $l = l(x, y)$, $(x, y) \in \mathcal{D}$; $l^2 = 1$ has the following properties:*

$$(3.3) \quad l_x^2 = l_y^2 = \nu > 0, \quad l_x l_y = 0, \quad \nu_x \nu_y \neq 0.$$

Conversely, if a unit vector function $l(x, y)$ has the properties (3.3), then there exists locally a unique (up to a motion) minimal strongly regular surface $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y)$ determined by

$$(3.4) \quad \mathbf{z}_x = -\frac{1}{\nu} l_x, \quad \mathbf{z}_y = \frac{1}{\nu} l_y,$$

so that (x, y) are canonical principal parameters and $\nu(x, y)$ is the normal curvature function of \mathcal{M} .

Proof: The equalities $l_x = -\nu \mathbf{z}_x$, $l_y = \nu \mathbf{z}_y$, and (2.3) imply (3.3).

For the inverse, it follows immediately that (3.3) implies (3.2). Therefore the system (3.4) is integrable and determines locally a surface $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y)$.

Since (3.4) and (3.2) imply (3.1), it follows that \mathcal{M} is a minimal strongly regular surface parameterized with canonical principal parameters, whose normal curvature function is $\nu = l_x^2 = l_y^2 > 0$.

Furthermore, it follows that the function $\nu = l_x^2 = l_y^2$ satisfies the equation (2.5). \square

Thus, any minimal strongly regular surface locally is determined by the system (3.4), where the unit vector function $l = l(x, y)$ satisfies the conditions (3.3). The so obtained minimal surface is parameterized with canonical principal parameters.

Let $S^2(1) : \xi^2 + \eta^2 + \zeta^2 = 1$ be the unit sphere centered at the origin O and $l(\xi, \eta, \zeta)$, $\zeta \neq 1$ be the position vector of an arbitrary point on $S^2(1)$, different from the pole $(0, 0, 1)$. The standard conformal parametrization of S^2 generated by the stereographic map

$$l : \begin{aligned} \xi &= \frac{2x}{x^2 + y^2 + 1}, \\ \eta &= \frac{2y}{x^2 + y^2 + 1}, \\ \zeta &= \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}, \end{aligned} \quad (x, y) \in \mathbb{R}^2$$

satisfies the equalities

$$l_x^2 = l_y^2 = \frac{4}{(x^2 + y^2 + 1)^2}, \quad l_x l_y = 0$$

and the unit vector function $l = l(x, y)$ generates the Enneper minimal surface parameterized with canonical principal parameters.

Now we make more precise the Weierstrass representation of minimal strongly regular surfaces in canonical principal parameters.

Proof of Theorem 1

Let $\mathcal{M} : \mathbf{z} = \mathbf{z}(x, y)$, $(x, y) \in \mathcal{D}$ be a minimal strongly regular surface parameterized with canonical principal parameters. Since the Gauss map $l = l(x, y)$ of \mathcal{M} is conformal and the

orthogonal frame field $l_u l_v l$ is left oriented, then the vector function l is given locally by the equalities

$$(3.5) \quad \begin{aligned} \xi &= \frac{2u(x, y)}{u^2(x, y) + v^2(x, y) + 1}, \\ l : \quad \eta &= \frac{2v(x, y)}{u^2(x, y) + v^2(x, y) + 1}, \\ \zeta &= \frac{u^2(x, u) + v^2(x, y) - 1}{u^2(x, y) + v^2(x, y) + 1}, \end{aligned}$$

where

$$w : \quad \begin{aligned} u &= u(x, y), & u_x &= v_y, \\ v &= v(x, y), & u_y &= -v_x \end{aligned}$$

is a holomorphic function in \mathbb{C} .

We denote

$$\mu := \frac{(u_x^2 + u_y^2)}{(u^2 + v^2 + 1)^2}.$$

If $\nu = \nu(x, y)$ is the normal curvature function of \mathcal{M} , then the vector function $\mathbf{z} = \mathbf{z}(x, y)$ satisfies the system

$$(3.6) \quad \begin{aligned} \mathbf{z}_x &= -\frac{1}{\nu} l_x = -\frac{1}{\nu} (u_x l_u - u_y l_v), \\ \mathbf{z}_y &= \frac{1}{\nu} l_y = \frac{1}{\nu} (u_y l_u + u_x l_v). \end{aligned}$$

It follows from (3.6) that $\nu = \frac{4(u_x^2 + u_y^2)}{(u^2 + v^2 + 1)^2} = 4\mu$. Hence the holomorphic function w satisfies the conditions

$$\mu > 0, \quad \mu_x \mu_y \neq 0.$$

Denoting by $w' = \frac{dw}{dz} = \frac{\partial w}{\partial z} = u_x - iu_y$, we find from (3.6)

$$(3.7) \quad \mathbf{z}_x - i\mathbf{z}_y = -\frac{1}{w'} \frac{(u^2 + v^2 + 1)^2}{4} (l_u + il_v).$$

Then (3.7) in view of (3.5) can be written in the form:

$$\begin{aligned} (z_1)_x - i(z_1)_y &= \frac{1}{2} \frac{w^2 - 1}{w'}, \\ (z_2)_x - i(z_2)_y &= -\frac{i}{2} \frac{w^2 + 1}{w'}, \\ (z_3)_x - i(z_3)_y &= -\frac{w}{w'}, \end{aligned}$$

which proves the assertion. \square

As an application we obtain a corollary for the solutions of the natural partial differential equation of minimal surfaces.

Corollary 3.2. *Any solution ν of the natural partial differential equation (2.5) satisfying the condition (2.6) locally is given by the formula*

$$(3.8) \quad \nu(x, y) = \frac{4(u_x^2 + u_y^2)}{(u^2 + v^2 + 1)^2}$$

where $w = u(x, y) + iv(x, y)$ is a holomorphic function in \mathbb{C} .

Proof: Let $\nu(x, y)$ be a solution to (2.5) satisfying the condition (2.6). Then the function ν generates locally a minimal strongly regular surface \mathcal{M} (unique up to a motion). According to Theorem 1 it follows that the normal curvature function ν of \mathcal{M} has locally the form (3.8). \square

It is a direct verification that any function ν given by (3.8), where $w = u + iv$ ($u_x^2 + u_y^2 > 0$) is a holomorphic function, satisfies (2.5).

Remark 3.3. The canonical Weierstrass representation is based on the Gauss map of the Enneper surface ($w = z$). It is clear that choosing the Gauss map of any other minimal strongly regular surface \mathcal{M} , we shall obtain its corresponding representation. This remark is also valid for the form (3.8) of the solutions of the natural partial differential equation (2.5).

REFERENCES

- [1] Ganchev G. and V. Mihova. *On the Invariant Theory of Weingarten Surfaces in Euclidean Space*. arXiv:0802.2191v1 [math.DG]

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